Combinations

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- Combinations are another basic counting principle.
- The most basic form of question that combinations address are as follows: If we have n objects, how many ways can we choose r of them (independently of order!)?

Ex: In bridge, you choose a hand of 13 cards (out of 52). The hand does not depend on the order.

If we just choose 13 cards, we have

$$\underline{52} \cdot \underline{51} \cdot \ldots \underline{40}. \quad \text{many options.}$$

which is <u>a lot</u>. <u>but</u> some of these have been double counted, since we have counted all the permutations of our hand as seperate hands. How many ways can we permute 13 cards? From last time: $13!$

So the number of possible hands is

$$\frac{52 \cdot 51 \cdot \ldots \cdot 40}{13!}$$

In general, given n object if we want to choose $0 \le r \le n$ of them, we can do this in

$$\frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!} := \underbrace{\binom{n}{r}}_{\substack{\text{"n choose r"} \\ \text{"nCr"}}}$$

$$\underbrace{\binom{n}{n} = \frac{n!}{(n-n)!\,n!} = 1, \quad \binom{n}{0} = \frac{n!}{n!\,0!} = 1.}_{\text{special cases.}}$$

Example ~~4~~ b): Given a group of 5 women and 7 men, How many ~~gro~~ commitees consisting of 2 women and 3 men can be chosen? What if 2 of the 7 men refuse to work together?

If there are no feuds, then from the group of women we have $\binom{5}{2} = 10$ possibilities, and for the men we have $\binom{7}{3} = 35$ possibilities. So, all together, there are $\binom{5}{2}\binom{7}{3} = 350$ possible commitees.

Now what about the feuding men?

There are $\binom{7}{3} = 35$ total ways to choose the men, but $\binom{2}{2}\binom{5}{1} = \binom{2}{2}\binom{7-2}{1} = 5$ options contain both the men.

So there are only $\binom{7}{3} - 5 = 30$ ways to choose the men. Hence there are $30 \cdot \underbrace{\binom{5}{2}}_{\text{women}} = 300$ total ways.

See text for more examples!!

The numbers $\binom{n}{r}$, $r \le n$, are called "binomial coefficients". This is because of the following fact

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Notice the symmetry $\binom{n}{k} = \binom{n}{n-k}$

The idea can be rephrased as follows:

"$\binom{n}{r}$ is the number of ways one can partition a set of size n into two sets: one of size k and the other of size $n-k$."

This illustrates the symmetry: partitioning a set of size n into a set of size k and $n-k$ is the same as partitioning the set into a set of size n.

Let's generalize this idea.

Suppose that $n_1 + \dots + n_r = n$.

How many ways can we partition a set of size n into r-many distinct subsets of size $n_1, n_2, \dots, n_r$?

Since the order of the groups doesn't matter, we get

- $\binom{n}{n_1}$ choices for the first group,
- $\binom{n-n_1}{n_2}$ choices for the second group.
- $\binom{n-n_1-n_2}{n_3}$ for the third group

$\vdots$

- $\binom{n-n_1-n_2-\dots-n_{r-1}}{n_r}$ for the last group.

So, all together:

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\cdots\binom{n-n_1-n_2-\dots-n_{r-1}}{n_r} = \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

check this!

We define $\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!\, n_2! \cdots n_r!}$

when $n = \sum_{i=1}^{r} n_i$.

These numbers are called multinomial coefficients, since we have:

The multinomial theorem:

$$(x_1 + \cdots + x_r)^n = \sum_{n_1 + \cdots + n_r = n} \binom{n}{n_1, \ldots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}$$

<u>Ex</u>: Twenty people, divided into teams of 5.

There are $\binom{20}{5,5,5,5} = 11{,}732{,}745{,}024$ (almost 12 billion.)

ways to partition 20 people into 4 groups of 5. But the order of the 4 teams does not matter, so we "cancel out" the extra permutations $(4!)$

So there are $\frac{20!}{4!\,5!\,5!\,5!\,5!}$ many such teams.

(See also 5b and 5c in the book.)